

Particle picture representation of the non-symmetric Rosenblatt process and Hermite processes of any order

Łukasz Treshczotko*
lukasz.treshczotko@gmail.com

March 6, 2017

Abstract

We provide a particle picture representation for the non-symmetric Rosenblatt process and for Hermite processes of any order, extending the result of Bojdecki, Gorostiza and Talarczyk in [3]. We show that these processes can be obtained as limits in the sense of finite-dimensional distributions of certain functionals of a system of particles evolving according to symmetric stable Lévy motions. In the case of k -Hermite processes the corresponding functional involves k -intersection local time of symmetric stable Lévy processes

Keywords: Rosenblatt process, Hermite processes, intersection local time, particle systems, stable Lévy processes, Wick product

2000 *Mathematics Subject Classification:* Primary:60G18 Secondary:60F17

1 Introduction

1.1 Hermite Processes and Generalized Hermite Processes

In this paper we study so called Hermite processes and their generalizations. A stochastic process $(X(t))_{t \geq 0}$ is said to be self-similar if there exists a constant $H > 0$ such that for any $a > 0$ $(X(at))_{t \geq 0} \stackrel{d}{=} (a^H X(t))_{t \geq 0}$, where the equality is in the sense of finite dimensional distributions. Any self-similar process which also has stationary increments is usually called H -sssi. H -sssi processes are studied mainly because they are the only possible limits of normalized partial sums of stationary sequences. To be more precise, Lamperti's theorem states that whenever $(X(n))_{n \in \mathbb{Z}}$ is a stationary sequence of random variables and

$$\frac{1}{A(N)} \sum_{n=1}^{[Nt]} X(n) \Rightarrow Y(t) \quad (1.1)$$

*Institute of Mathematics, University of Warsaw, Banacha 2 02-097 Warsaw

in the sense of finite dimensional distributions, where $A(N) \rightarrow \infty$ as $N \rightarrow \infty$, then $(Y(t))_{t \geq 0}$ must be an H -sssi process. H is usually called the *Hurst coefficient*. It is exactly in this setting that Hermite processes arose in the first place, in the so called *non-central limit theorems* (see [5]). Following [8] we briefly sketch it. Let $(\xi_n)_{n \in \mathbb{Z}}$ be a centered stationary Gaussian sequence with variance equal to 1 such that

$$r(n) := \mathbb{E}(\xi_n \xi_0) = n^{\frac{2H-2}{k}} L(n), \quad (1.2)$$

with $H \in (\frac{1}{2}, 1)$, $k \geq 1$ and L - a function slowly varying at infinity. Take any function $g : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\mathbb{E}g(\xi_0) = 0$ and $\mathbb{E}g(\xi_0)^2 < \infty$, which has the following expansion in Hermite polynomials:

$$g(x) = \sum_{j=0}^{\infty} c_j H_j(x), \quad (1.3)$$

where H_j is the j -th Hermite polynomial, $c_j = \frac{1}{j!} \mathbb{E}(g(\xi_0) H_j(\xi_0))$ and k is the smallest j with $c_j \neq 0$. If we introduce the following sequence of stochastic processes:

$$Z_H^{k,n}(t) := \frac{1}{n^H} \sum_{l=1}^{[nt]} g(\xi_l), \quad n \in \mathbb{N}, t \geq 0, \quad (1.4)$$

then

$$Z_H^{k,n} \xrightarrow{d} c_k Z_H^k, \quad (1.5)$$

where Z_H^k is the k -Hermite process and the convergence holds in the sense of finite-dimensional distributions (from now on we will use the notation \xrightarrow{d} to denote this type of convergence). This is how Hermite processes were obtained in the first place and investigating the convergence of partial sums is still a useful way of obtaining new stochastic processes.

Hermite processes may also be described with the help of multiple Wiener-Itô integrals. For an introduction to these integrals see [11]. For $k \in \mathbb{N}$ (using the notation from [2]) one can represent a k -Hermite process as

$$Z_H^k(t) := a_{k,d} \int_{\mathbb{R}^k}' \int_0^t \prod_{j=1}^k (s - x_j)_+^{d-1} ds W(dx_1) \dots W(dx_k), \quad (1.6)$$

where W is a two-sided Brownian motion, $\frac{1}{2}(1 - \frac{1}{k}) < d < \frac{1}{2}$ is the number satisfying $H = kd - k/2 + 1$ so that $1/2 < H < 1$, $a_{k,d}$ is a positive constant chosen so that $\mathbf{Var}(Z_H^k(1)) = 1$ and "'' above the integral sign indicates that the diagonal is excluded from integration. For our purposes it will be convenient to use a so called *spectral representation* which uses multiple Wiener-Itô integrals as defined in [10]:

$$Z_H^k(t) = c_{k,d} \int_{\mathbb{R}^k}'' \frac{e^{i(u_1 + \dots + u_k)t} - 1}{i(u_1 + \dots + u_k)} |u_1|^{-d} \dots |u_k|^{-d} \widehat{W}(du_1) \dots \widehat{W}(du_k); \quad (1.7)$$

here the constant $c_{k,d}$ serves the same purpose as $a_{k,d}$ in (1.6) and \widehat{W} is the random complex Gaussian white noise measure on \mathbb{R} , where d is given as before. For more about this representation and random spectral measures (which are used extensively in this paper) see Chapter 3 of [10], for a brief overview see Subsection 2.2.

k -Hermite processes “live” in the k -th Wiener chaos. For $k = 1$ the 1-Hermite process is just a fractional Brownian motion and the 2-Hermite process is called the *Rosenblatt process*. For a comprehensive introduction to the Rosenblatt process see [13]. For all $k \in \mathbb{N}$ k -Hermite processes have the same covariance given by

$$\mathbb{E}(Z_H^k(s)Z_H^k(t)) =: R(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}).$$

It was not at first obvious whether or not Hermite processes were the only self-similar processes with stationary increments in their respective Wiener chaoses. The only sssi-process in the first Wiener chaos is the fractional Brownian motion (see Theorem 1.3.3 in [6]). This is not true for Wiener chaoses of order $k \geq 2$ and the first example of this fact was the *non-symmetric Rosenblatt process* (see [9] and [14] for a nice introduction to this process), which is obtained if we replace the kernel $\prod_{j=1}^k (s - x_j)_+^{d-1}$ in equation (1.6) (for $k = 2$) by

$$g(x,y) = x^{-1+\alpha/2}y^{-1+\beta/2}\mathbf{1}_{\{x>0,y>0\}}, \quad (1.8)$$

with $\alpha, \beta \in (0,1)$ and $\alpha + \beta > 1$. Even in this relatively simple case with $k = 2$, $\alpha, \alpha', \beta, \beta' \in (0,1)$ and $\alpha + \beta = \alpha' + \beta' > 1$ the corresponding non-symmetric processes have different laws for different choices of α, α', β and β' (see Proposition 3.10 in [14]). More generally, the initial kernel can be replaced by even more general functions to obtain the so called *generalized Hermite processes* introduced and investigated in [2].

1.2 Representation of Hermite Processes

Hermite processes arise naturally as limits of normalized partial sums of stationary sequences as in (1.4). Recently in [3] a different type of limit theorem was proved. It was shown that the Rosenblatt process can be obtained from a Poisson system of particles evolving according to α -stable processes. Our aim is to extend this representation to the general k -Hermite processes and the non-symmetric Rosenblatt process. Let us briefly sketch the particle system we are going to use.

Let (x^j) be a Poisson system with Lebesgue intensity measure on \mathbb{R} and let $(\xi^j)_{j=1}^\infty$ be independent symmetric α -stable Levy processes with the index of stability $\alpha \in (0,1)$. Notice that we only consider the values of the parameter α for which $(\xi_t)_{t \geq 0}$ is transient. We also assume that these processes are independent of the points (x^j) . In the end, assume that $(\sigma_j)_{j=1}^\infty$ are i.i.d random

variables such that $\mathbb{P}(\sigma_1 = 1) = \mathbb{P}(\sigma_1 = -1) = \frac{1}{2}$ and that these variables are independent of everything else. The particle system is given by $(x^j + \xi_t^j)_{t \geq 0}$. Thus the initial position of the particles is given by the points (x^j) and they evolve independently according to the symmetric α -stable processes. Furthermore, we independently assign charges σ_j to these particles. This is the underlying system which will give rise to the stochastic processes studied in this paper and will be present throughout our work.

In [3] it was shown that the process given by

$$\xi_t^T = \frac{1}{T} \sum_{j \neq k} \sigma_j \sigma_k \langle \Lambda(x^j + \xi_t^j, x^k + \xi_t^k; T), \mathbf{1}_{[0, t]} \rangle, \quad t \geq 0, \quad (1.9)$$

where Λ is the *intersection local time* of two independent α -stable Lévy processes (see section 2.3 in [3], [1] and section 2.1 in this paper), converges, as $T \rightarrow \infty$, (up to a constant) for $\alpha \in (1/2, 1)$, in $C([0, \tau])$ for $\tau \in (0, \infty)$, to the Rosenblatt process with the Hurst coefficient $H = \alpha$. We will show how a non-symmetric Rosenblatt process is obtained from the same particle system. We will also extend the result of [3] to k -Hermite processes for $k \geq 3$.

1.3 Results

First we will state a limit theorem leading to a non-symmetric Rosenblatt process. Consider the particle system described in Section 1.2 and let $\beta > \alpha$. Define

$$\eta_t^T = \frac{1}{T} \sum_{j \neq k} \sigma_j \sigma_k \int_0^T \int_0^T \mathbf{1}_{[0, t]}(x^j + \xi_r^j) \frac{1}{|x^k + \xi_s^k - x^j - \xi_r^j|^{1 - \frac{\beta - \alpha}{2}}} dr ds \quad (1.10)$$

for $T > 0, t \geq 0$. The fact that the above functional is well defined (in the sense that the sum in (1.10) converges in $L^2(\Omega)$) will be shown in Subsection 3.1. The first of the two main results in this paper is the following theorem.

Theorem 1. *Let α and β be such that $1 > \beta > \alpha > 0$, $\alpha + \beta > 1$. Then, as $T \rightarrow \infty$, the processes $(\eta_t^T)_{t \geq 0}$ converge in the sense of finite dimensional distributions to the non-symmetric Rosenblatt process with parameters (α, β) , up to a multiplicative constant.*

The quite explicit formulation of the approximating process in Theorem 1 is what makes it particularly appealing.

The second question we set out to answer was whether the representation given by (1.9) can be extended to k -Hermite processes for general $k \geq 2$. To formulate our result we must first introduce the so called *k-intersection local time* (k -ILT), which is an extension of the notion of *intersection local time* (ILT) and was first considered in [12]. Informally k -ILT of cadlag processes ρ^1, \dots, ρ^k

at time $T \geq 0$ (denoted here by $\Lambda^{(k)}$) can be defined by

$$\begin{aligned} \langle \Lambda^{(k)}(\rho^1, \dots, \rho^k; T), \phi \rangle &= \\ &= \int_{[0, T]^k} \phi(\rho_{s_1}^1) \delta_0(\rho_{s_2}^2 - \rho_{s_1}^1) \dots \delta_0(\rho_{s_k}^k - \rho_{s_1}^1) ds_1 \dots ds_k, \end{aligned} \quad (1.11)$$

where δ_0 is the Dirac distribution at 0 and $\phi \in \mathcal{S}$ (the Schwartz space of rapidly decreasing function). One gives a meaning to (1.11) by approximating δ_0 by smooth fuctions. The precise definition and the proof of existence of ILT in the case of independent symmetric α -stable Lévy processes is given in Section 4.1, which is an extension of Proposition 5.1 in [4]. The answer to the second question is provided by the following theorem.

Theorem 2. *Let $k \geq 2$ be a natural number and $\alpha \in (1 - \frac{1}{k}, 1)$. For $T > 0$ we denote*

$$\rho_t^T := \frac{1}{T^{k/2}} \sum_{j_1 \neq \dots \neq j_k} \sigma_{j_1} \dots \sigma_{j_k} \langle \Lambda^{(k)}(x^{j_1} + \xi^{j_1}, \dots, x^{j_k} + \xi^{j_k}; T), \mathbf{1}_{[0, t]} \rangle, t \geq 0. \quad (1.12)$$

Then, as $T \rightarrow \infty$, the process $(\rho_t^T)_{t \geq 0}$ converges in the sense of finite-dimensional distributions to the k -Hermite process Z_H^k with the Hurst coefficient equal to $H = 1 - (1 - \alpha)k/2$.

The main scheme of the proofs of Theorems 1 and 2 is similar to the one employed in [3], in particular the idea of using Wick products of an appropriate \mathcal{S}' -valued random variable. To stress some of the main differences and difficulties that had to be overcome in our case let us point out that in the case of Theorem 1 it was at first not at all clear what functional of a particle system can be used to approximate the non-symmetric Rosenblatt process. Also, since the functional is different, we need to use different approximations.

In case of Theorem 2 it was more or less clear that one should use (1.12) as the approximating process. However, due to the fact that now we have to deal with k -intersection local times and Wick products of order k there are some non-trivial technical difficulties (see the proof of (4.16)).

Additionally, in the proof of the representation of the symmetric Rosenblatt process in [3] the identification of the limiting distribution was done using the cumulants and the fact that the finite-dimensional distributions of this process are determined by its moments. In our paper we will take a different route and utilize the Itô formula for multiple Wiener-Itô integrals (Theorem 4.3 in [10]).

The paper is organized as follows. In Section 2 we fix the notation and introduce some of the concepts which will be later used extensively to prove the results. Section 3 contains proof of Theorem 1 and in Section 4 we discuss the existence of k -intersection local time and prove Theorem 2.

2 Notation and Background

2.1 Notation

Throughout the rest of the paper, by $\mathcal{S}(\mathbb{R}^d)$ we will denote the Schwartz space of *real-valued* smooth rapidly decreasing functions on \mathbb{R}^d , $d \in \mathbb{N}$, $\mathcal{S}'(\mathbb{R}^d)$ will be the space tempered distributions. Let \mathcal{F} denote the class of non-negative symmetric, infinitely differentiable functions on \mathbb{R} with support in $B(0, 1) = \{x \in \mathbb{R} : |x| < 1\}$ satisfying $\int_{\mathbb{R}} f(x) dx = 1$. These functions will be used to to approximate Dirac delta distributions. For any $f \in \mathcal{F}$, $\epsilon > 0$ put

$$f_{\epsilon}(x) = \epsilon^{-d} f\left(\frac{x}{\epsilon}\right), \quad x \in \mathbb{R}.$$

We will also use the following definition of the Fourier transform. For $\phi \in \mathcal{S}(\mathbb{R}^d)$

$$\widehat{\phi}(x) := \int_{\mathbb{R}^d} e^{ixz} \phi(z) dz, \quad x \in \mathbb{R}^d.$$

Throughout the paper λ_1 will denote one-dimensional Lebesgue measure on \mathbb{R} and \Rightarrow will denote convergence in law.

Since we are interested in the convergence of stochastic processes in the sense of finite dimensional distributions it is convenient to introduce the following class of functions. Let \mathcal{A} be the family of functions of the form

$$\psi = \sum_{j=1}^m a_j \mathbf{1}_{I_j}, \quad (2.1)$$

where $a_j \in \mathbb{R}$ and I_j is a bounded interval for each $j = 1, \dots, m$. For $g \in \mathcal{F}$ and $\psi \in \mathcal{A}$ we will write $\psi_{\kappa} := \psi * g_{\kappa}$, without explicitly referring to the function g to make the notation more transparent. As it happens, we will always require that in the limit the particular choice of g is irrelevant as far as our purposes are concerned. Notice that

$$\widehat{\psi_{\kappa}}(x) = \widehat{\psi}(x) \widehat{g_{\kappa}}(x) = \widehat{\psi}(x) \widehat{g}(\kappa x), \quad (2.2)$$

so that

$$|\widehat{\psi_{\kappa}}(x)| \leq |\widehat{\psi}(x)|, \quad x \in \mathbb{R}, \quad (2.3)$$

since $|\widehat{g}(z)| \leq 1$ for all $z \in \mathbb{R}$.

2.2 Generalized Gaussian Random Fields and Fractional Brownian Motion

One of the important tools which we will be using in this paper are \mathcal{S}' -valued random variables. In particular, we will be working with centered Gaussian \mathcal{S}' -valued random variables. For each $\alpha < 1$ there exists a centered Gaussian \mathcal{S}' -random variable X with covariance functional given by

$$\mathbb{E}\langle X, \phi \rangle \langle X, \psi \rangle = \frac{1}{\pi} \int_{\mathbb{R}} \widehat{\phi}(x) \overline{\widehat{\psi}(x)} |x|^{-\alpha} dx, \quad \phi, \psi \in \mathcal{S}(\mathbb{R}), \quad (2.4)$$

The spectral measure of this field is given by $G(dx) = |x|^{-\alpha}dx$. As noted in [3], for $\alpha \in (0, 1)$, X can be approximated by the normalized total charge occupation of our particle system from Subsection 1.2 on the interval $[0, T]$, that is by a functional given by

$$\langle X_T, \phi \rangle = \frac{1}{\sqrt{T}} \sum_j \sigma_j \int_0^T \phi(x_j + \xi_s^j) ds, \quad \phi \in \mathcal{S}(\mathbb{R}). \quad (2.5)$$

By an L^2 extension we may evaluate X on functions from a much wider class than $\mathcal{S}(\mathbb{R})$ and in fact $(\langle X, \mathbf{1}_{[0,t]} \rangle)_{t \geq 0}$ is up to a constant the fractional Brownian motion with Hurst coefficient equal to $H = \frac{1+\alpha}{2}$. The particle system we are working with can be used to approximate this process as in the theorem below.

Theorem 3 (Theorem 2.1 in [3]). *For $\alpha \in (0, 1)$, as $T \rightarrow \infty$, we have:*

- (i) $X_T \Rightarrow X$ in $\mathcal{S}'(\mathbb{R})$,
- (ii) $(\langle X_T, \mathbf{1}_{[0,t]} \rangle)_{t \geq 0}$ converges in the sense of finite dimensional distributions to $(KB_t^H)_{t \geq 0}$ where K is a constant and B^H is a fractional Brownian motion with Hurst coefficient $H = \frac{1+\alpha}{2}$.

Remark 4. *The random field $(\langle X, \phi \rangle)_{\phi \in \mathcal{S}(\mathbb{R})}$ can be used (see chapter 3 in [10] for details) to construct a random spectral measure Z_G associated with this field such that $\langle X, \phi \rangle = \int_{\mathbb{R}} \hat{\phi}(x) Z_G(dx)$ for $\phi \in \mathcal{S}$. We will use it extensively throughout this paper.*

3 Non-symmetric Rosenblatt Process

3.1 Preliminaries

Let us denote

$$\langle \Delta(x + \xi^1, y + \xi^2; T), \phi \rangle = \int_0^T \int_0^T \phi(x + \xi_r^1) |y + \xi_s^2 - x - \xi_r^1|^{\frac{\beta-\alpha}{2}-1} dr ds, \quad (3.1)$$

where ξ^1, ξ^2 are independent symmetric α -stable Lévy processes. According to Lemma 15 in the Appendix to show that (3.1) is well defined (in the sense that the series converges in $L^2(\Omega)$) it suffices to show that for any $t > 0$ and independent symmetric α -stable processes ξ^1, ξ^2 we have

$$\langle \Delta(\cdot + \xi^1, \cdot + \xi^2; T), \mathbf{1}_{[0,t]} \rangle \in L^2(\mathbb{R} \times \mathbb{R} \times \Omega, \lambda_1 \otimes \lambda_1 \otimes \mathbb{P}), \quad (3.2)$$

which is done in the following lemma.

Lemma 5. *Let $\alpha, \beta \in (0, 1)$ be such that $\alpha + \beta > 1$, $\beta > \alpha$ and let $\phi \in L^1(\mathbb{R})$ be bounded. Then*

$$I := \mathbb{E} \left(\int_{\mathbb{R}^2} \left(\int_0^T \int_0^T \phi(x + \xi_r^1) |y + \xi_s^2 - x - \xi_r^1|^{\gamma-1} ds dr \right)^2 dx dy \right) \leq CT^2 < \infty, \quad (3.3)$$

where $\gamma := \frac{\beta-\alpha}{2}$ and C is a constant.

Proof. Without loss of generality we may assume that $\phi \geq 0$. We have

$$\begin{aligned} I &= \mathbb{E} \int_{\mathbb{R}^2} \int_{[0,T]^4} \phi(x + \xi_r^1) |y + \xi_s^2 - x - \xi_r^1|^{\gamma-1} \\ &\quad \phi(x + \xi_u^1) |y + \xi_v^2 - x - \xi_u^1|^{\gamma-1} dr ds du dv dx dy \end{aligned} \quad (3.4)$$

$$\begin{aligned} &= \mathbb{E} \int_{\mathbb{R}^2} \int_{[0,T]^4} \phi(x) |y - x|^{\gamma-1} \phi(x + \xi_u^1 - \xi_r^1) \\ &\quad |y + \xi_v^2 - \xi_s^2 - x - (\xi_u^1 - \xi_r^1)|^{\gamma-1} dr ds du dv dx dy \end{aligned} \quad (3.5)$$

$$\begin{aligned} &= \int_{\mathbb{R}^4} \int_{[0,T]^4} \phi(x) |y - x|^{\gamma-1} \phi(z) |w - z|^{\gamma-1} \\ &\quad \phi(z) |w - z|^{\gamma-1} p_{|u-r|}(z - x) p_{|v-s|}(w - y) dr du ds dv dx dy dz dw, \end{aligned} \quad (3.6)$$

where p is the α -stable transition density. The second equality uses Fubini theorem. Since for $x \neq 0$ $\int_0^\infty p_s(x) ds = C|x|^{\alpha-1}$ for some constant $C = C(\alpha)$ and

$$\int_0^T \int_0^T p_{|u-r|}(x) dr du \leq \frac{2CT}{|x|^{1-\alpha}}, \quad (3.7)$$

we see that I can be bounded by

$$C^2 T^2 4 \int_{\mathbb{R}^2} \phi(x) |y - x|^{\gamma-1} \phi(z) |w - z|^{\gamma-1} \frac{1}{|z - x|^{1-\alpha}} \frac{1}{|w - y|^{1-\alpha}} dx dy dz dw. \quad (3.8)$$

Notice that

$$\int_{\mathbb{R}} |y - x|^{\gamma-1} |w - y|^{\alpha-1} dy = C'(\alpha, \beta) |w - x|^{\frac{\alpha+\beta}{2}-1}, \quad (3.9)$$

with $C'(\alpha, \beta)$ being a constant. Similarly

$$\int_{\mathbb{R}} |w - x|^{\frac{\alpha+\beta}{2}-1} |w - z|^{\gamma-1} dw = C''(\alpha, \beta) |z - x|^{\beta-1}. \quad (3.10)$$

Thus

$$I \leq C'''(\alpha, \beta) T^2 \int_{\mathbb{R}^2} \phi(x) \phi(z) |z - x|^{\alpha+\beta-2}. \quad (3.11)$$

The right-hand side of (3.11) is finite since ϕ is bounded and in $L^1(\mathbb{R})$ and $\alpha + \beta > 1$. \square

Let us briefly discuss the main ideas behind the proof of convergence of finite-dimensional distributions of η^T to those of the Rosenblatt process. We will show that the functional η_t^T is close to $\langle : X_T \otimes X_T :, \Phi_t \rangle$, where $: X_T \otimes X_T :$ is the Wick product of the process X_T defined by (2.5) and $\Phi_t(x, y) = \mathbf{1}_{[0,t]}(x) |x - y|^{\frac{\beta-\alpha}{2}-1}$, $x, y \in \mathbb{R}$. Recall that the Wick product $: X_T \otimes X_T :$ is defined in the following way. First, for $\Phi \in \mathcal{S}(\mathbb{R}^2)$ of the form

$$\Phi = \sum_{j=1}^m \phi_j \otimes \psi_j, \quad (3.12)$$

where $\phi_j, \psi_j \in \mathcal{S}$, for $j = 1, \dots, m$, we set

$$\langle : X_T \otimes X_T :, \Phi \rangle := \sum_{j=1}^m \left(\langle X_T, \phi_j \rangle \langle X_T, \psi_j \rangle - \mathbb{E}(\langle X_T, \phi_j \rangle \langle X_T, \psi_j \rangle) \right). \quad (3.13)$$

(3.13) can then be extended to arbitrary $\Phi \in \mathcal{S}(\mathbb{R}^2)$. Next we would like to use Theorem 3 to obtain that $\langle : X_T \otimes X_T :, \Phi_t \rangle$ converges, as $T \Rightarrow \infty$, in distribution to $\langle : X \otimes X :, \Phi_t \rangle$. Finally we will show that $(\langle : X \otimes X :, \Phi_t \rangle)_{t \geq 0}$ is up to a constant a non-symmetric Rosenblatt process. One of the difficulties lies in the fact that Φ_t is not in $\mathcal{S}(\mathbb{R}^2)$ and we must approximate it by functions from the Schwartz space. $\langle : X \otimes X :, \Phi_t \rangle$ is then understood as a limit under these approximations.

We now proceed to discussing our approximating functions and some of their properties. For convenience let us set $\gamma = \frac{\beta - \alpha}{2}$. We are going to approximate the function $y \mapsto |y|^{\gamma-1}$ by the convolution $\int_{\mathbb{R}} |y - z|^{\gamma-1} f_{\epsilon}(z) dz$ (where $f \in \mathcal{F}$ and $\epsilon > 0$), and then use the fact that as $\epsilon \rightarrow 0$ the integral converges (up to a constant depending only on γ) to $|y|^{-\gamma}$. However, the function $y \mapsto \int_{\mathbb{R}} |y - z|^{\gamma-1} f_{\epsilon}(z) dz$ still does not belong to $\mathcal{S}(\mathbb{R})$ as it vanishes slowly. To overcome this obstacle take $\delta \in (0, 1)$, let $h_{\delta}(x) := |x|^{\gamma-1} \mathbf{1}_{\delta < |x| < \frac{1}{\delta}}$ and put

$$V^{\delta} \phi(x) := \int_{\mathbb{R}} h_{\delta}(x - y) \phi(y) dy, \quad \phi \in \mathcal{S}, x \in \mathbb{R}. \quad (3.14)$$

We approximate the function $y \mapsto |y|^{\gamma-1}$ by $V^{\delta} f_{\epsilon}$. Let us also define $V\phi(x) := \lim_{\delta \rightarrow 0+} V^{\delta} \phi(x)$. This limit exists as long as $\int_{\mathbb{R}} |x - y|^{\gamma-1} |\phi(y)| dy < \infty$. It is easy to see that for $f \in C_c^{\infty}(\mathbb{R})$, $V^{\delta} f \in C_c^{\infty}(\mathbb{R})$ for any $\delta \in (0, 1)$. The Fourier transform of $V^{\delta} \phi$ is given by

$$\widehat{V^{\delta} \phi}(x) = \widehat{h_{\delta}}(x) \widehat{\phi}(x), \quad x \in \mathbb{R}, \quad (3.15)$$

for $\phi \in \mathcal{S}$. In the sequel we will need several of properties of operators V^{δ} . We list them in the lemma below.

Lemma 6. *Let f be in \mathcal{F} . The operators V^{δ} defined by 3.14 have the following properties:*

(i) *for each $\delta \in (0, 1)$ $V^{\delta} f$ is nondecreasing as $\delta \searrow 0$ and*

$$Vf(x) = \lim_{\delta \rightarrow 0+} V^{\delta} f(x); \quad (3.16)$$

(ii)

$$\lim_{\epsilon \rightarrow 0+} Vf_{\epsilon}(x) = |x|^{\gamma-1}, \quad x \neq 0; \quad (3.17)$$

(iii) *for each $\epsilon > 0$*

$$Vf_{\epsilon}(x) \leq \|f\|_{\infty} \frac{2^{2-\gamma}}{\gamma} |x|^{\gamma-1}. \quad (3.18)$$

Proof. Parts (i) and (ii) are obvious. To prove part (iii) fix $\epsilon > 0$ and consider two cases. Assume first that $|x| \geq 2\epsilon$. Then $Vf_\epsilon(x) \leq 1/|x-\epsilon|^{1-\gamma} \leq 2^{1-\gamma}|x|^{\gamma-1}$. If $|x| < 2\epsilon$ then

$$Vf_\epsilon(x) = \int_{-\epsilon}^{\epsilon} \frac{1}{|x-y|^{1-\gamma}} \frac{1}{\epsilon} f\left(\frac{y}{\epsilon}\right) dy \leq \|f\|_\infty \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \frac{1}{|y|^{1-\gamma}} dy = \|f\|_\infty \frac{2^{2-\gamma}}{\gamma} |x|^{\gamma-1}, \quad (3.19)$$

and this finishes the proof. \square

Now we can define the approximating functional which will be at the center of our investigation. Let $\epsilon, \delta \in (0, 1)$, $f \in \mathcal{F}$. Mimicking (3.1), for any $\phi \in \mathcal{S}(\mathbb{R})$ and a pair of real-valued cadlag processes η, ξ we put

$$\langle \Delta_{\epsilon, \delta}^f(\eta, \xi; T), \phi \rangle := \int_0^T \int_0^T \phi(\eta_u) (V^\delta f_\epsilon)(\xi_v - \eta_u) dudv. \quad (3.20)$$

For $\phi \geq 0$ the above integral converges pointwise as $\delta \rightarrow 0$ and then $\epsilon \rightarrow 0$ by Lemma 6 (to a possibly infinite limit) independently of the choice of f . This limit is given by

$$\langle \Delta(\eta, \xi; T), \phi \rangle = \int_0^T \int_0^T \phi(\eta_u) |\xi_v - \eta_u|^{\gamma-1} dudv, \quad \phi \in \mathcal{S}. \quad (3.21)$$

Now we proceed to show that in the setting that interests us most ($\eta = x + \xi^1$, $\xi = y + \xi^2$ for independent symmetric α -stable Lévy processes ξ^1, ξ^2 and $x, y \in \mathbb{R}$) the random variables given by (3.20) and (3.21) are meaningful. We can only show that $\Delta(\eta_1, \eta_2; T)$ exists as an \mathcal{S}' -valued random variable in the setting in which $\alpha, \beta \in (1/2, 1)$. The convergence of $\langle \Delta_{\epsilon, \delta}^f(\eta_1, \eta_2; T), \phi \rangle$ in $L^2(\Omega)$ for fixed $x, y \in \mathbb{R}$ remains an open question. However, we will only need the following.

Lemma 7. *Let ξ^1, ξ^2 be independent symmetric α -stable processes with $\alpha \in (0, 1)$.*

(i) *For every $\phi \in \mathcal{S}(\mathbb{R})$, $T > 0$ the function given by*

$$(x, y, \omega) \mapsto \langle \Delta_{\epsilon, \delta}^f(x + \xi^1, y + \xi^2; T), \phi \rangle \quad (3.22)$$

converges in $L^2(\mathbb{R}^2 \times \Omega, \lambda_1 \otimes \lambda_1 \otimes \mathbb{P})$, as ϵ, δ go to 0, to the function

$$(x, y, \omega) \mapsto \Delta(x, y, \omega) = \int_0^T \int_0^T \phi(x + \xi_r^1(\omega)) |y + \xi_s^2(\omega) - x - \xi_r^2(\omega)|^{\frac{\beta-\alpha}{2}-1} dr ds. \quad (3.23)$$

(ii) *The L^2 -convergence as in (i) holds also if we replace ϕ by any function $\psi \in \mathcal{A}$. Moreover, for ϕ of the form $\phi = \psi_\kappa = \psi * g_\kappa$, where $\kappa \in (0, 1)$ and $g \in \mathcal{F}$, the convergence is uniform in κ .*

Here λ_1 is the one dimensional Lebesgue measure and \mathbb{P} is the underlying probability measure.

Proof of Lemma 7. The proof is quite straightforward once we have established Lemma 5. We have

$$\begin{aligned}
& \mathbb{E} \int_{\mathbb{R}^2} \left(\langle \Delta_{\epsilon, \delta}^f(x + \xi^1, y + \xi^2; T), \phi \rangle - \langle \Delta(x + \xi^1, y + \xi^2; T), \phi \rangle \right)^2 dx dy \\
& \leq \mathbb{E} \int_{\mathbb{R}^2} \left(\int_{[0, T]^2} |\phi(x + \xi_u^1)| \left| V_{f_\epsilon}^\delta(y + \xi_v^2 - x - \xi_u^1) \right. \right. \\
& \quad \left. \left. - |y + \xi_v^2 - x - \xi_u^1|^{\gamma-1} \right| dudv \right)^2 dx dy \\
& \leq \mathbb{E} \int_{\mathbb{R}^2} \left(\int_{[0, T]^2} |\phi(x + \xi_u^1)| C_1(f, \gamma) |y + \xi_v^2 - x - \xi_u^1|^{\gamma-1} dudv \right)^2 dx dy, \quad (3.24)
\end{aligned}$$

which is finite by Lemma 5. In the second inequality in (3.24) we have used part (iii) of Lemma 6. Now, using parts (i) and (ii) of the same Lemma and dominated convergence theorem we get the desired convergence. Evidently, the particular choice of f from \mathcal{F} is irrelevant. The rest of the proof is now straightforward. \square

3.2 Proof of Theorem 1

Proof of Theorem 1. The proof of Theorem 1 more or less follows the line of reasoning of the proof of Theorem 3.5 in [3]. We will study the behavior of the following functional, which approximates the functional in the statement of Theorem 1:

$$\eta_{f, \phi, \epsilon, \delta}^T := \frac{1}{T} \sum_{j \neq k} \sigma_j \sigma_k \langle \Delta_{\epsilon, \delta}^f(x^j + \xi^j, x^k + \xi^k; T), \phi \rangle, \quad (3.25)$$

where, as before $\epsilon, \delta \in (0, 1)$, $f \in \mathcal{F}$, $\phi \in \mathcal{S}(\mathbb{R})$, $T > 0$. It is well defined by Lemma 15. Moreover, the above functional converges in $L^2(\Omega)$, as $\epsilon, \delta \rightarrow 0$ (uniformly in $T \geq 1$ and independently of the choice of $f \in \mathcal{F}$) to a random variable η_ϕ^T given by

$$\eta_\phi^T := \frac{1}{T} \sum_{j \neq k} \sigma_j \sigma_k \langle \Delta(x^j + \xi^j, x^k + \xi^k; T), \phi \rangle, \quad (3.26)$$

which again follows from Lemmas 7 and 15. Recall that $\gamma = (\beta - \alpha)/2$. Let ψ be any function from \mathcal{A} and $\Psi_{\epsilon, \delta, \phi}^f(x, y) := \phi(x)(V^\delta f_\epsilon)(y - x)$, where V^δ is defined by (3.14). We will show the convergence, as $T \rightarrow \infty$, of the characteristic function of η_{T, ψ_κ} to the characteristic function of the finite-dimensional distributions of the non-symmetric Rosenblatt process. Using inequality $|\mathbb{E}(e^{iX}) - \mathbb{E}(e^{i\tilde{X}})| \leq 2\mathbb{E}|X - \tilde{X}| \leq 2 \left(\mathbb{E}|X - \tilde{X}|^2 \right)^{\frac{1}{2}}$ (for real valued random variables X and \tilde{X} , it is enough to show that:

$$\lim_{\kappa \rightarrow 0} \sup_{T \geq 1} \mathbb{E} |\eta_{\psi}^T - \eta_{\psi_{\kappa}}^T|^2 = 0, \quad (3.27)$$

$$\lim_{\epsilon, \delta \rightarrow 0} \sup_{T \geq 1} \sup_{\kappa \in (0,1)} \mathbb{E} |\eta_{f, \psi_{\kappa}, \epsilon, \delta}^T - \eta_{\psi_{\kappa}}^T|^2 = 0, \quad (3.28)$$

$$\lim_{T \rightarrow \infty} \sup_{\kappa \in (0,1)} \mathbb{E} \left| \langle X_T \otimes X_T :, \Psi_{\epsilon, \delta, \phi}^f \rangle - \eta_{f, \phi, \epsilon, \delta}^T \right|^2 = 0, \quad \epsilon, \delta > 0, \phi \in \mathcal{S}, \quad (3.29)$$

$$\langle X_T \otimes X_T :, \Psi_{\epsilon, \delta, \phi}^f \rangle \xrightarrow{T \rightarrow \infty} \langle X \otimes X :, \Psi_{\epsilon, \delta, \phi}^f \rangle, \quad \epsilon, \delta > 0, \phi \in \mathcal{S}, \quad (3.30)$$

$$\lim_{\epsilon, \delta, \kappa \rightarrow \infty} \mathbb{E} \left| \langle X \otimes X :, \Psi_{\epsilon, \delta, \psi_{\kappa}}^f \rangle - \int_{\mathbb{R}^2} \widehat{\psi}(x+y) |y|^{-\gamma} Z_G(dx) Z_G(dy) \right|^2 = 0, \quad (3.31)$$

where Z_G is the random spectral measure as in Remark 4.

Similarly as in the proof of Lemma 7 it is easy to show (using dominated convergence theorem) that (3.27) holds. It is enough to notice that $|\psi_{\kappa}| \leq |\psi|$. Lemma 7 and Lemma 8.1 from [4] give us (3.28) (for details see the proof of equation (6.26) in [3]). The proof of (3.29) is very similar to the proof of equation (6.27) in [3] so we will only sketch it. Recalling (2.5) we may write

$$\begin{aligned} \langle X_T \otimes X_T :, \Psi_{\epsilon, \delta, \phi}^f \rangle &= \frac{1}{T} \int_{[0, T]^2} \left(\sum_{j, k} \sigma_j \sigma_k \Psi_{\epsilon, \delta, \phi}^f(x^j + \xi_s^j, x^k + \xi_u^k) \right. \\ &\quad \left. - \int_{\mathbb{R}} \mathbb{E} \Psi_{\epsilon, \delta, \phi}^f(x + \xi_s^1, x + \xi_u^1) dx \right) ds du \\ &= \eta_{T, \phi, \epsilon, \delta}^f + \frac{1}{T} \int_{[0, T]^2} \left(\sum_j \Psi_{\epsilon, \delta, \phi}^f(x^j + \xi_s^j, x^j + \xi_u^j) \right. \\ &\quad \left. - \int_{\mathbb{R}} \mathbb{E} \Psi_{\epsilon, \delta, \phi}^f(x + \xi_s^1, x + \xi_u^1) dx \right) ds du. \end{aligned} \quad (3.32)$$

This implies that, again by Lemma 14 in the Appendix,

$$\begin{aligned} \mathbb{E} \left| \langle X_T \otimes X_T :, \Psi_{\epsilon, \delta, \phi}^f \rangle - \eta_{T, \phi, \epsilon, \delta}^f \right|^2 &= \\ \frac{1}{T} \int_{[0, T]^4} \int_{\mathbb{R}} \mathbb{E} \Psi_{\epsilon, \delta, \phi}^f(x + \xi_s^1, x + \xi_u^1) \Psi_{\epsilon, \delta, \phi}^f(x + \xi_r^1, x + \xi_v^1) dx ds du dr dv. \end{aligned} \quad (3.33)$$

The rest of the argument is exactly the same as in the proof of equation (6.26) in [3] because $V^{\delta} f_{\epsilon} \in \mathcal{S}(\mathbb{R})$ implies that $\Psi_{\epsilon, \delta, \phi}^f$ is in $\mathcal{S}(\mathbb{R}^2)$. The convergence in (3.30) follows from Lemma 6.3 in [3]. For the proof of (3.31) we will look

at Hermite processes from the point of view of multiple Wiener-Itô integrals. From Theorem 4.7 in [10] it follows that for any $\Phi \in \mathcal{S}(\mathbb{R}^2)$

$$\langle : X \otimes X :, \Phi \rangle = \int_{\mathbb{R}^2}'' \widehat{\Phi}(x, y) Z_G(dx) Z_G(dy), \quad (3.34)$$

where Z_G is the random spectral measure, as in Remark 4, corresponding to the spectral measure $G(dx) = |x|^{-\alpha} dx$. Hence, using (3.15),

$$\langle : X \otimes X :, \Psi_{\epsilon, \delta, \psi_\kappa}^f \rangle = \frac{1}{2\pi} \int_{\mathbb{R}^2}'' \widehat{\psi_\kappa}(x+y) \widehat{f_\epsilon}(y) \widehat{h_\delta}(y) Z_G(dx) Z_G(dy). \quad (3.35)$$

By dominated convergence and L^2 isometry $\langle : X \otimes X :, \Psi_{\epsilon, \delta, \psi_\kappa}^f \rangle$ converges in $L^2(\Omega)$ as $\epsilon, \delta, \kappa \rightarrow 0$ to $\frac{1}{2\pi} \int_{\mathbb{R}^2}'' \widehat{\psi}(x+y) |y|^{-\gamma} Z_G(dx) Z_G(dy)$, which by the change of variables formula for multiple Wiener-Itô integrals (Theorem 4.4 in [10]), is equal to

$$\frac{1}{2\pi} \int_{\mathbb{R}^2}'' \widehat{\psi}(x+y) |x|^{-\frac{\alpha}{2}} |y|^{-\frac{\beta}{2}} \widehat{W}(dx) \widehat{W}(dy), \quad (3.36)$$

where \widehat{W} is a complex-valued Fourier transform of white noise (see discussion after equation (2.6) in [3]). When we replace ψ with $\mathbf{1}_{[0,t]}$, then (using definition 3.8 in [2] and spectral representation discussed in the following remarks) we can define

$$Z_t^T := \eta_{T, \mathbf{1}_{[0,t]}}, \quad t \geq 0, \quad (3.37)$$

then $(Z_t^T)_{t \geq 0}$ converges (up to a constant), as $T \rightarrow \infty$, in the sense of finite dimensional distributions, to the non-symmetric (α, β) -Rosenblatt process. \square

4 Representation of Hermite Processes

4.1 k -intersection local time of independent α -stable processes

Following [4] we would like to extend the definition of intersection local time to k -intersection local time. For $\epsilon > 0, \phi \in \mathcal{S}(\mathbb{R}), f \in \mathcal{F}$ and an integer $k \geq 2$ put

$$\Phi_{\epsilon, \phi}^f(x_1, \dots, x_k) := \phi(x_1) f_\epsilon(x_2 - x_1) \dots f_\epsilon(x_k - x_1). \quad (4.1)$$

Note that $\Phi_{\epsilon, \phi}^f \in \mathcal{S}(\mathbb{R}^k)$ and

$$\widehat{\Phi_{\epsilon, \phi}^f}(x_1, \dots, x_k) = \widehat{\phi}(x_1 + \dots + x_k) \widehat{f_\epsilon}(x_2) \dots \widehat{f_\epsilon}(x_k). \quad (4.2)$$

Using (4.1) we can define the *approximate intersection local time* of k real valued cadlag stochastic processes. For any cadlag processes ρ_1, \dots, ρ_k taking values

in \mathbb{R} , $\phi \in \mathcal{S}(\mathbb{R})$ and $f \in \mathcal{F}$ we denote the approximate intersection local time at time $T > 0$ by

$$\begin{aligned} \langle \Lambda_\epsilon^f(\rho^1, \dots, \rho^k; T), \phi \rangle &= \\ &= \int_{[0, T]^k} \phi(\rho_{s_1}^1) f_\epsilon(\rho_{s_2}^2 - \rho_{s_1}^1) \dots f_\epsilon(\rho_{s_k}^k - \rho_{s_1}^1) ds_1 \dots ds_k. \end{aligned} \quad (4.3)$$

Definition 8. *If there exists an \mathcal{S}' -valued random variable $\Lambda^{(k)}(\rho_1, \dots, \rho_k)$ such that for each $\phi \in \mathcal{S}$ and $f \in \mathcal{F}$ $\langle \Lambda_\epsilon^f(\rho^1, \dots, \rho^k; T), \phi \rangle$ converges to $\langle \Lambda^{(k)}(\rho_1, \dots, \rho_k), \phi \rangle$ in $L^2(\Omega)$ and the limit is independent of the choice of $f \in \mathcal{F}$ then $\Lambda^{(k)}$ is called the k -intersection local time of ρ_1, \dots, ρ_k .*

We have the following extension of Proposition 5.1 in [4].

Lemma 9. *Let ξ_1, \dots, ξ_k be independent α -stable Lévy processes with $\alpha \in (1 - \frac{1}{k}, 1)$. Then for any starting points $x_1, \dots, x_k \in \mathbb{R}$ and $\phi \in \mathcal{S}$ the k -intersection local time $\langle \Lambda^{(k)}(x_1 + \xi^1, \dots, x_k + \xi^k; T), \phi \rangle$ exists. Moreover $\Lambda^{(k)}$ can be evaluated for any function ϕ in \mathcal{A} .*

Proof. To prove the lemma it is enough to show that for any $f, g \in \mathcal{F}$, $x_1, \dots, x_k \in \mathbb{R}$ and each $\phi \in \mathcal{S}$, the limit

$$\lim_{\epsilon, \delta \rightarrow 0} \mathbb{E} \left(\langle \Lambda_\epsilon^f(x_1 + \xi^1, \dots, x_k + \xi^k; T), \phi \rangle \langle \Lambda_\delta^g(x_1 + \xi^1, \dots, x_k + \xi^k; T), \phi \rangle \right) \quad (4.4)$$

exists and does not depend on the choice of f and g . The proof is at first very similar to the proof of Proposition 3.3 and 5.1 in [4]. Writing out the expectation in (4.4) using the α -stable transition densities, passing to the Fourier transform, using Plancherel formula and then using the estimate

$$\int_{[0, T]^2} |\widehat{\mu}_{s,u}(z, z')| ds du \leq C(T) \frac{1}{1 + |z + z'|^\alpha} \left(\frac{1}{1 + |z|^\alpha} + \frac{1}{1 + |z'|^\alpha} \right),$$

where $C(T)$ is a constant and $\mu_{s,u}$ is the law of (ξ_s^1, ξ_u^1) , the proof is reduced to showing that

$$\begin{aligned} I &= \int_{\mathbb{R}^{2k}} |\widehat{\phi}(z_1 + \dots + z_k) \widehat{\phi}(w_1 + \dots + w_k)| \\ &\quad \times b(z_1, w_1) \dots b(z_k, w_k) dz_1 \dots dz_k dw_1 \dots dw_k < \infty, \end{aligned} \quad (4.5)$$

where $b(z, w) = C(T) \frac{1}{1 + |z + w|^\alpha} \left(\frac{1}{1 + |z|^\alpha} + \frac{1}{1 + |w|^\alpha} \right)$. To show that we will use Hölder inequality. First, fix some $\lambda \in (0, 1)$. Rewrite equation (4.5) as follows.

$$\begin{aligned} I &= \int_{\mathbb{R}^{2k}} |\widehat{\phi}(z_1 + \dots + z_k)^{\frac{k}{k}} \widehat{\phi}(w_1 + \dots + w_k)^{\frac{k}{k}}| \\ &\quad \times b(z_1, w_1)^{\lambda \frac{k}{k}} \dots b(z_k, w_k)^{\lambda \frac{k}{k}} \\ &\quad \times b(z_1, w_1)^{(1-\lambda) \frac{k-1}{k-1}} \dots b(z_k, w_k)^{(1-\lambda) \frac{k-1}{k-1}} dz_1 \dots dz_k dw_1 \dots dw_k. \end{aligned} \quad (4.6)$$

The integrand can be written as $g_1(\mathbf{z}, \mathbf{w}) \dots g_k(\mathbf{z}, \mathbf{w})$, where

$$\begin{aligned} g_j(\mathbf{z}, \mathbf{w}) &= |\widehat{\phi}(z_1 + \dots + z_k) \widehat{\phi}(w_1 + \dots + w_k)|^{\frac{1}{k}} h(\mathbf{z}, \mathbf{w})^{\frac{\lambda}{k}} \\ &\quad \times b(z_1, w_1)^{(1-\lambda)\frac{1}{k-1}} \dots b(z_{j-1}, w_{j-1})^{(1-\lambda)\frac{1}{k-1}} \\ &\quad \times b(z_{j+1}, w_{j+1})^{(1-\lambda)\frac{1}{k-1}} \dots b(z_k, w_k)^{(1-\lambda)\frac{1}{k-1}}, \end{aligned} \quad (4.7)$$

and $h(\mathbf{z}, \mathbf{w}) = b(z_1, w_1) \dots b(z_k, w_k)$. By Hölder inequality

$$I \leq \prod_{j=1}^k \left(\int_{\mathbb{R}^{2k}} g_j(\mathbf{z}, \mathbf{w})^k d\mathbf{z} d\mathbf{w} \right)^{\frac{1}{k}} \quad (4.8)$$

For $\phi \in \mathcal{S}$ or \mathcal{A} , $|\widehat{\phi}(x)| \leq \frac{C}{1+|x|}$, $x \in \mathbb{R}$, where C is a constant. Now, taking λ close enough to 0, we have $\frac{(1-\lambda)k\alpha}{k-1} > 1$. This implies that each factor in (4.8) is finite. \square

In fact we will not need this “pointwise” sort of convergence and we will only utilize a weaker result (which is an analogue of Lemma 7) to be able to formulate the main theorem of this section in a rigorous way.

Lemma 10. *Assume that $\alpha \in (1 - \frac{1}{k}, 1)$. Then $\langle \Lambda_\epsilon^f(\cdot + \xi^1, \dots, \cdot + \xi^k; T), \phi \rangle$ converges in $L^2(\mathbb{R}^k \times \Omega, \lambda_k \otimes \mathbb{P})$ as $\epsilon \rightarrow 0$ for any $\phi \in \mathcal{S}(\mathbb{R})$ and the limit is independent of the choice of $f \in \mathcal{F}$. Moreover, if we replace ϕ by any function of the form ψ_κ as in (2.1), the convergence is uniform in $\kappa \in (0, 1)$. We also denote this limit by $\langle \Lambda^{(k)}(\cdot + \xi^1, \dots, \cdot + \xi^k; T), \phi \rangle$.*

The proof of this lemma is similar to the case when $k = 2$ and amounts to showing that that

$$\int_{\mathbb{R}^k} |\widehat{\phi}(x_1 + \dots + x_k)|^2 |x_1|^{-\alpha} \dots |x_k|^{-\alpha} dx_1 \dots dx_k < \infty, \quad (4.9)$$

for $\phi \in \mathcal{S}(\mathbb{R})$ or \mathcal{A} . In the symmetric case above (4.9) follows by Hölder inequality, similarly as in Lemma 9. Indeed, putting

$$s_j(x_1, \dots, x_k) := \prod_{i=1, i \neq j}^k \left(\frac{1}{1 + |x_i|^\alpha} \right)^{1/(k-1)}$$

$j = 1, \dots, k$, we get

$$\begin{aligned} &\int_{\mathbb{R}^k} |\widehat{\phi}(x_1 + \dots + x_k)|^2 \frac{1}{1 + |x_1|^\alpha} \dots \frac{1}{1 + |x_k|^\alpha} dx_1 \dots dx_k \leq \\ &\leq \prod_{j=1}^k \left(\int_{\mathbb{R}^k} |\widehat{\phi}(x_1 + \dots + x_k)|^2 s_j(x_1, \dots, x_k)^k dx_1 \dots dx_k \right)^{\frac{1}{k}}, \end{aligned} \quad (4.10)$$

which is finite since, by assumption, $\alpha > 1 - \frac{1}{k}$.

Remark 11. In fact one can prove that for any choice of $\alpha_1, \dots, \alpha_k \in (0, 1)$ satisfying $\alpha_1 + \dots + \alpha_k > k - 1$ and any $f \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}^k} |\widehat{f}(x_1 + \dots + x_k)|^2 |x_1|^{-\alpha_1} \dots |x_k|^{-\alpha_k} dx_1 \dots dx_k < \infty, \quad (4.11)$$

but the proof is a little more complicated.

In order to use the properties of \mathcal{S}' -valued random variables we introduce the following approximating functional:

$$\rho_{f, \epsilon, \phi}^T := \frac{1}{T^{k/2}} \sum_{j_1 \neq \dots \neq j_k} \sigma_{j_1} \dots \sigma_{j_k} \langle \Lambda_\epsilon^f(x^{j_1} + \xi^{j_1}, \dots, x^{j_k} + \xi^{j_k}; T), \phi \rangle, \quad (4.12)$$

where $T > 0, \epsilon > 0, f \in \mathcal{F}, \phi \in \mathcal{S}(\mathbb{R})$. It is well defined by Lemmas 15 and 10. The same Lemmas also show that the functional given by

$$\rho_{\psi_\kappa}^T := \frac{1}{T^{k/2}} \sum_{j_1 \neq \dots \neq j_k} \sigma_{j_1} \dots \sigma_{j_k} \langle \Lambda^{(k)}(x^{j_1} + \xi^{j_1}, \dots, x^{j_k} + \xi^{j_k}; T), \psi_\kappa \rangle, \quad (4.13)$$

is well defined for any $\kappa \in (0, 1), T > 0, \psi \in \mathcal{A}$ and is an $L^2(\Omega)$ -limit of the functional in 4.12 with ϕ replaced by ψ_κ .

4.2 Proof of the representation

Proof of Theorem 2. The proof follows the footsteps of the proof of Theorem 3.5 in [3] with some necessary generalizations. From now on we fix $\alpha \in (1 - \frac{1}{k}, 1)$ and $f \in \mathcal{F}$. We are going to prove the following claims:

$$\lim_{\kappa \rightarrow 0} \sup_{T \geq 1} \mathbb{E} |\rho_\psi^T - \rho_{\psi_\kappa}^T|^2 = 0, \quad (4.14)$$

$$\lim_{\epsilon \rightarrow 0} \sup_{T \geq 1} \sup_{\kappa \in (0, 1)} \mathbb{E} |\rho_{\psi_\kappa}^T - \rho_{f, \epsilon, \psi_\kappa}^T|^2 = 0, \quad (4.15)$$

$$\lim_{T \rightarrow \infty} \sup_{\kappa \in (0, 1)} \mathbb{E} \left| \langle : X_T \otimes \dots \otimes X_T :, \Phi_{\epsilon, \psi_\kappa}^f \rangle - \rho_{f, \epsilon, \psi_\kappa}^T \right|^2 = 0, \quad \epsilon > 0, \quad (4.16)$$

$$\langle : X_T \otimes \dots \otimes X_T :, \Phi_{\epsilon, \psi_\kappa}^f \rangle \Rightarrow \langle : X \otimes \dots \otimes X :, \Phi_{\epsilon, \psi_\kappa}^f \rangle, \quad \epsilon > 0, \kappa > 0, \quad (4.17)$$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \sup_{\kappa \in (0, 1)} \mathbb{E} \left| \int_{\mathbb{R}^k} \widehat{\psi_\kappa}(x_1 + \dots + x_k) Z_G(dx_1) \dots Z_G(dx_k) \right. \\ \left. - \langle : X \otimes \dots \otimes X :, \Phi_{\epsilon, \psi_\kappa}^f \rangle \right|^2 = 0, \end{aligned} \quad (4.18)$$

$$\lim_{\kappa \rightarrow 0} \mathbb{E} \left| \int_{\mathbb{R}^k}'' \widehat{\psi}_\kappa(x_1 + \dots + x_k) Z_G(dx_1) \dots Z_G(dx_k) - \int_{\mathbb{R}^k}'' \widehat{\psi}(x_1 + \dots + x_k) Z_G(dx_1) \dots Z_G(dx_k) \right|^2 = 0. \quad (4.19)$$

Here $: Z \otimes \dots \otimes Z :$ stands for the k -th Wick product of the random variable Z . For definition see equation (4.22). From Lemma 15 we have (with F replaced by $\langle (\Lambda_\epsilon^f - \Lambda), \psi_\kappa \rangle$) the following inequality:

$$\begin{aligned} \mathbb{E} |\rho_{\psi_\kappa}^T - \rho_{f, \epsilon, \psi_\kappa}^T|^2 &\leq \frac{2k!}{T^k} \int_{\mathbb{R}^k} \mathbb{E} |\langle (\Lambda_\epsilon^f - \Lambda^{(k)})(x_1 + \xi^1, \dots, x_k + \xi^k; T), \psi_\kappa \rangle|^2 \\ &\leq 2k! 2^k \int_{\mathbb{R}^k} |\widehat{\psi}_\kappa(x_1 + \dots + x_k)|^2 |\widehat{f}_\epsilon(x_1 + \dots + x_k) - 1|^2 \\ &\quad \times |x_1|^{-\alpha} \dots |x_k|^{-\alpha} dx_1 \dots dx_k. \end{aligned} \quad (4.20)$$

By (4.9), dominated convergence theorem and the fact that $\widehat{\psi}_\kappa(z) \leq \widehat{\psi}(z)$ for $z \in \mathbb{R}$ we get (4.15). The proof of (4.14) is very similar and we skip it.

The hardest part is to prove (4.16). From [3] we know that (iii) holds for $k = 2$. Let Φ be of the form

$$\Phi = \sum_{j=1}^m \phi^{(1,j)} \otimes \dots \otimes \phi^{(k,j)}, \quad (4.21)$$

where each $\phi^{(s,t)}$ is in $\mathcal{S}(\mathbb{R})$ for $s = 1, \dots, k$, $t = 1, \dots, m$. By definition

$$\begin{aligned} \langle : X_T \otimes \dots \otimes X_T :, \Phi \rangle = \\ \sum_{j=1}^m \sum_{A \in \mathcal{M}} (-1)^{|A|} \prod_{\{s,t\} \in A} \mathbb{E}(\langle X_T, \phi^{(s,j)} \rangle) \mathbb{E}(\langle X_T, \phi^{(t,j)} \rangle) \prod_{n \notin \cup A} \langle X_T, \phi^{(n,j)} \rangle, \end{aligned} \quad (4.22)$$

where \mathcal{M} is the set of unordered pairs $\{s, t\} \subset \{1, \dots, k\}$, such that all the elements in these pairs are distinct. In particular $|\cup A| = 2|A|$. The sum above is over all distinct sets A of this form including the empty set. If we define the approximating functional by

$$\rho_\Phi^T := \frac{1}{T^{k/2}} \sum_{j_1 \neq \dots \neq j_k} \sigma_{j_1} \dots \sigma_{j_k} \int_{[0, T]^k} \Phi(x^{j_1} + \xi_{s_1}^{j_1}, \dots, x^{j_k} + \xi_{s_k}^{j_k}) ds_1 \dots ds_k, \quad (4.23)$$

then one can easily see that

$$\mathbb{E}(\langle : X_T \otimes \dots \otimes X_T :, \Phi \rangle - \rho_\Phi^T)^2 = \mathbb{E}(\langle : X_T \otimes \dots \otimes X_T :, \Phi \rangle^2) - \mathbb{E}(\rho_\Phi^T)^2. \quad (4.24)$$

This follows from the fact that in ρ_Φ^T we have summation over distinct indices $\{j_1, \dots, j_k\} \in \mathbb{N}$ and so the only nonzero terms in $\mathbb{E}(\langle : X_T \otimes \dots \otimes X_T :, \Phi \rangle \rho_\Phi^T)$

are those that correspond to $A = \emptyset$ in (4.22). Furthermore, if we recall the sum in (2.5) defining $\langle X_T, \phi \rangle$ for $\phi \in \mathcal{S}(\mathbb{R})$, then it is obvious that $\mathbb{E}(\langle X_T \otimes \dots \otimes X_T : \Phi \rangle \rho_\Phi^T) = \mathbb{E}(\rho_\Phi^T)^2$. Let us denote $\mathbb{E}(\langle X_T \otimes \dots \otimes X_T : \Phi \rangle^2)$ by I . Then

$$\begin{aligned} I &= \sum_{j=1}^m \sum_{j'=1}^m \sum_{A, A' \in \mathcal{M}} (-1)^{|A|} (-1)^{|A'|} \prod_{(s,t) \in A} \mathbb{E} \langle X_T, \phi^{(s,j)} \rangle \langle X_T, \phi^{(t,j)} \rangle \\ &\quad \times \prod_{(s',t') \in A'} \mathbb{E} \langle X_T, \phi^{(s',j')} \rangle \langle X_T, \phi^{(t',j')} \rangle \\ &\quad \times \mathbb{E} \left(\prod_{n \notin \cup A} \langle X_T, \phi^{(n,j)} \rangle \prod_{n' \notin \cup A'} \langle X_T, \phi^{(n',j')} \rangle \right). \end{aligned} \quad (4.25)$$

Computing the last expected value in (4.25) amounts to summation over different choices of the *diagonals* just as in the proof of Lemma 15. To illustrate it consider first the case $A = \emptyset = A'$. Then, we have no covariances and are left with

$$\begin{aligned} I_\emptyset &= \mathbb{E} \left(\sum_{j=1}^m \sum_{j'=1}^m \langle X_T, \phi^{(1,j)} \rangle \dots \langle X_T, \phi^{(k,j)} \rangle \right. \\ &\quad \times \langle X_T, \phi^{(1,j')} \rangle \dots \langle X_T, \phi^{(k,j')} \rangle \Big) \\ &= \mathbb{E} \left(\sum_{\substack{j_1, \dots, j_k \\ j_{k+1}, \dots, j_{2k}}} \sigma_{j_1} \dots \sigma_{j_k} \sigma_{j_{k+1}} \dots \sigma_{j_{2k}} F_T(x^{j_1} + \xi^{j_1}, \dots, x^{j_k} + \xi^{j_k}) \right. \\ &\quad \times F_T(x^{j_{k+1}} + \xi^{j_{k+1}}, \dots, x^{j_{2k}} + \xi^{j_{2k}}), \Big) \end{aligned} \quad (4.26)$$

where

$$\begin{aligned} F_T(x_1 + \xi^1, \dots, x_k + \xi^k) &:= \\ \frac{1}{T^{k/2}} \int_{[0,T]^k} \Phi(x_1 + \xi_{s_1}^1, \dots, x_k + \xi_{s_k}^k) ds_1 \dots ds_k, \quad x_1, \dots, x_k \in \mathbb{R}. \end{aligned} \quad (4.27)$$

The only terms in the sum in (4.26), whose expected values are non-zero, are those for which for every $l \in \{j_1, \dots, j_k, j_{k+1}, \dots, j_{2k}\}$ there is an *even* number of indices taking that value. This sum can be split into a finite number of sums over different *diagonals*. To be precise, by a *diagonal* \mathcal{C} we mean a partition of $\{1, 2, \dots, 2k\}$ into a disjoint family of subsets C_1, \dots, C_m of $\{1, 2, \dots, 2k\}$ such that $|C_l|$ is even for $l = 1, \dots, m$. Then the term in (4.26) corresponding to this diagonal is given by

$$\begin{aligned} &\sum_{\substack{j_{v_1^1} = \dots = j_{v_{k_1}^1}, \dots, v_1^1, \dots, v_{k_1}^1 \in C_1 \\ \dots \\ j_{v_1^m} = \dots = j_{v_{k_m}^m}, \dots, v_1^m, \dots, v_{k_m}^m \in C_m}} \sigma_{j_1} \dots \sigma_{j_k} \sigma_{j_{k+1}} \dots \sigma_{j_{2k}} \\ &\quad \times F_T(x^{j_1} + \xi^{j_1}, \dots, x^{j_k} + \xi^{j_k}) F_T(x^{j_{k+1}} + \xi^{j_{k+1}}, \dots, x^{j_{2k}} + \xi^{j_{2k}}). \end{aligned} \quad (4.28)$$

Now, a diagonal \mathcal{C} is *large* if any of the sets C_1, \dots, C_m has more than two elements. In due course we will see that the sums over these diagonals behave as $\frac{1}{T}$ as $T \rightarrow \infty$. All other diagonals are *pairings* of different charges. This means that for these diagonals all C_l 's have exactly two elements. We will say that a pairing is *normal* if for every $C \in \mathcal{C}$, C has exactly one element from $\{1, \dots, k\}$ and one element from $\{k+1, \dots, 2k\}$. All other non-large diagonals will be called *non-normal pairings*. Notice that the choice of A, A' in (4.25) corresponds to fixing some particular part of the diagonal over which summation is being done. Looking at (4.25) from this perspective, there will only be normal pairings in the sums corresponding to $A, A' = \emptyset$. We can use the same notation for sums that will emerge from the term $\mathbb{E} \left(\prod_{n \notin \cup A} \langle X_T, \phi^{(n,j)} \rangle \prod_{n' \notin \cup A'} \langle X_T, \phi^{(n',j')} \rangle \right)$ in (4.25). Putting all this together we can write

$$\mathbb{E}(\langle X_T \otimes \dots \otimes X_T, \Phi \rangle^2) = I_1 + I_2 + R, \quad (4.29)$$

where in I_1 we have only the sums over the diagonals which correspond to normal pairings. We see immediately that $I_1 = \mathbb{E} \eta_{T,\Phi}^2$. I_2 corresponds to the sums over non-large non-normal pairings (notice that all the sums in (4.25) with $A \neq \emptyset$ or $A' \neq \emptyset$ will be in I_2) and R contains only sums over large diagonals. Notice that the terms in (4.29) may be written with the help of the function F_T given by (4.27) and extend continuously to general $\Phi \in \mathcal{S}(\mathbb{R}^2)$, not necessarily of the form (4.21).

We are going to show that $I_2 = 0$. Fix a non-large non-normal pairing \mathcal{C} . Assume that in the fixed diagonal over which the summation is being performed there are n non-normal pairs B formed between members of the sequence (j_1, \dots, j_k) and n non-normal pairs B' formed between members of the sequence (j_{k+1}, \dots, j_{2k}) . All the other pairs (there are $k - 2n$ of them) are normal. The sum over our fixed diagonal is going to appear in terms from (4.25) with $|A|, |A'| \leq n$. Fix $c, d \leq n$ and consider the summands in I_2 for which $|A| = c, |A'| = d$. The sum over \mathcal{C} is going to appear in exactly $\binom{n}{c} \binom{n}{d}$ summands with $c + d = n$ with the sign equal to $(-1)^{c+d}$. This can be justified as follows. The sum over \mathcal{C} can only appear in the terms with $A \subset B$ and $A' \subset B'$ and there will be $\binom{n}{c}$ choices of A with $A \subset B$ and $\binom{n}{d}$ choices of A' with $A' \subset B'$. We see that for each $0 \leq m \leq 2n$ with $m = |A| + |A'|$ the sum corresponding to our fixed diagonal will appear exactly $\sum_{l=0}^m \binom{n}{m-l} \binom{n}{l} = \binom{2n}{m}$ with sign equal to $(-1)^m$. Hence, the number of times (with signs taken into account) the sum over our fixed diagonal will appear in I_2 is exactly $\sum_{m=0}^{2n} (-1)^m \binom{2n}{m} = 0$. This proves that $I_2 = 0$.

R can be split into a finite number of sums over large diagonals, all of which have the property that the summation is taken over indices $(j_1, \dots, j_k, j_{k+1}, \dots, j_{2k})$ such that at least four of them are equal. To finish the proof of (4.16) we fix $\epsilon \in (0, 1)$ and take $\Phi = \Phi_{\epsilon, \psi_\kappa}^f$ (see (4.1)). It remains to show that $R = R_{T, \Phi_{\epsilon, \psi_\kappa}^f}$ converges to 0 as $T \rightarrow \infty$ uniformly in $\kappa \in (0, 1)$. Put $\theta(x) := \frac{1}{1+|x|^2}, x \in \mathbb{R}$. Notice that

$$|\Phi(x_1, \dots, x_k)| \leq p(\Phi) \frac{1}{1+|x_1|^2} \cdots \frac{1}{1+|x_k|^2} = p(\Phi) \theta(x_1) \cdots \theta(x_k), \quad (4.30)$$

where $p(\Phi)$ is a continuous seminorm on $\mathcal{S}(\mathbb{R}^k)$, given by

$$p(\Phi) = \sup_{x_1 \in \mathbb{R}, \dots, x_k \in \mathbb{R}} |(1 + |x_1|^2) \dots (1 + |x_k|^2) \Phi(x_1, \dots, x_k)|. \quad (4.31)$$

Thanks to this

$$\sup_{\kappa \in (0,1)} p(\Phi_{\epsilon, \psi_\kappa}^f) \leq C(\epsilon, f), \quad (4.32)$$

with $C(\epsilon, f)$ being a constant depending only on f and ϵ and independent of κ . To fix our attention, let us consider an example of a large diagonal with $k = 3$. This diagonal is given by requiring that $j_1 = j_2 = j_5 = j_6$ and $j_3 = j_4$. Then the expected value of the sum corresponding to this diagonal is given by

$$\frac{1}{T^{\frac{3}{2}}} \int_{\mathbb{R}^2} \mathbb{E} (F_T(x_1 + \xi^1, x_1 + \xi^1, x_2 + \xi^2) F_T(x_2 + \xi^2, x_1 + \xi^1, x_1 + \xi^1)) dx_1 dx_2. \quad (4.33)$$

The absolute value of the above integral is no bigger than

$$\begin{aligned} \frac{1}{T^{\frac{3}{2}}} p(\Phi)^2 \int_{\mathbb{R}^2} \int_{[0,T]^6} \mathbb{E} \Big(\theta(x_1 + \xi_{u_1}^1) \theta(x_1 + \xi_{u_2}^1) \theta(x_2 + \xi_{u_3}^2) \\ \times \theta(x_2 + \xi_{u_4}^2) \theta(x_1 + \xi_{u_5}^1) \theta(x_1 + \xi_{u_6}^1) \Big) du_1 \dots du_6 dx_1 dx_2. \end{aligned} \quad (4.34)$$

By (4.32), for $\Phi = \Phi_{\epsilon, \psi_\kappa}^f$, the integral in (4.34) can be bounded uniformly in κ by an integral which (by independence) can be written as a product of two integrals times a constant $C(\epsilon, f)$. One of the factors of this product (the one corresponding to the pairing $j_3 = j_4$) is bounded by a constant. The other is given by

$$\frac{1}{T^2} \int_{[0,T]^4} \int_{\mathbb{R}} \mathbb{E} (\theta(x + \xi_s^1) \theta(x + \xi_u^1) \theta(x + \xi_r^1) \theta(x + \xi_v^1)) dx ds du dr dv.$$

Following the proof of Theorem 3.5 in [3] we see that the above is bounded by

$$\frac{1}{T^2} \int_0^T \int_{\mathbb{R}} \theta(x) U(\theta U(\theta U \theta))(x) dx ds \leq \frac{1}{T} C_2, \quad (4.35)$$

where C_2 is another constant, and U is the potential of an α -stable semigroup. The second inequality above follows from the fact that $U\psi$ is bounded. In the case of blocks larger than four the argument is very similar. To conclude, $R \leq \frac{C}{T} p(\Phi)^2$, where C is a constant, which depends only on k, ϵ and f (the bound $\frac{1}{T}$ was given in [3] only for the diagonal with the largest element consisting of four equal indexes, but having larger diagonals is even better which can easily be inferred from the proof of equation (6.27) in [3]). This means that we can write $\mathbb{E} (\langle : X_T \otimes \dots \otimes X_T :, \Phi \rangle - \rho_{T, \Phi})^2 \leq \frac{C}{T} p(\Phi)^2$ for $\Phi \in \mathcal{S}(\mathbb{R}^k)$.

To prove (4.17) we will need the following generalization of lemma 6.3 in [3]

Lemma 12. *Let $(X_T)_{T \geq 1}$ be a family of \mathcal{S}' -valued random variables such that*

$$\sup_{T \geq 1} \mathbb{E} \langle X_T, \phi \rangle^2 \leq p^2(\phi), \quad \phi \in \mathcal{S}(\mathbb{R}),$$

for some continuous Hilbertian seminorm p on $\mathcal{S}(\mathbb{R})$. Suppose that $X_T \Rightarrow X$ and $\mathbb{E} \langle X_T, \phi \rangle^2 \rightarrow \mathbb{E} \langle X, \phi \rangle^2$ for $\phi \in \mathcal{S}(\mathbb{R})$ as $T \rightarrow \infty$. Then $X_T \otimes \dots \otimes X_T$ and $X \otimes \dots \otimes X$ are well defined and $X_T \otimes \dots \otimes X_T \Rightarrow X \otimes \dots \otimes X$ as $T \rightarrow \infty$.

This together with Theorem 3 implies (4.17). We proceed to prove (4.18) and (4.19). Fix $k \in \mathbb{N}, k \geq 2$. Let $\alpha \in (1 - \frac{1}{k}, 1)$ and let $(X_\phi)_{\phi \in \mathcal{S}(\mathbb{R})}$ be a generalized centered Gaussian random field over the Schwartz space with spectral measure $G(dx) = |x|^{-\alpha} dx$. Notice that, as before, using Theorem 4.7 in [10], we might write

$$\langle : X \otimes \dots \otimes X :, \Phi \rangle = \int_{\mathbb{R}^k}'' \widehat{\Phi}(x_1, \dots, x_k) Z_G(dx_1) \dots Z_G(dx_k), \quad (4.36)$$

for any $\Phi \in \mathcal{S}(\mathbb{R}^k)$. Whenever $\int_{\mathbb{R}^k}'' |\widehat{\Phi}(x_1 + \dots + x_k)|^2 G(dx_1) \dots G(dx_k) < \infty$, we see that by dominated convergence theorem,

$$\langle : X \otimes \dots \otimes X :, \Phi_{\epsilon, \phi}^f \rangle \xrightarrow{L^2(\Omega)} \int_{\mathbb{R}^k}'' \widehat{\Phi}(x_1 + \dots + x_k) Z_G(dx_1) \dots Z_G(dx_k). \quad (4.37)$$

Given the above (4.18) and (4.19) follow immediately. Establishing (4.14) - (4.19) shows that the finite-dimensional distributions of $(\eta_t^T)_{t \geq 0}$ converge to the finite dimensional distributions of the k -Hermite process given by (1.7) \square

Remark 13. *In fact we have shown that for $\Phi \in \mathcal{S}(\mathbb{R}^k)$ the functional defined by*

$$\rho_{T, \Phi} := \frac{1}{T^{k/2}} \sum_{j_1 \neq \dots \neq j_k} \sigma_{j_1} \dots \sigma_{j_k} \int_{[0, T]^k} \Phi(x^{j_1} + \xi_{s_1}^{j_1}, \dots, x^{j_k} + \xi_{s_k}^{j_k}) ds_1 \dots ds_k, \quad (4.38)$$

converges (up to a constant) in distribution to

$$\int_{\mathbb{R}^k}'' \widehat{\Phi}(z_1, \dots, z_k) Z_G(dz_1) \dots Z_G(dz_k), \quad (4.39)$$

as $T \rightarrow \infty$.

A Appendix

We would like to have the generalizations of some of the facts used in [4] and [3]. First we state the generalized version of Lemma 8.1 from [4], which is the simplified version of the so called Mecke-Palm formula (see for instance equation (2.10) in [7]).

Lemma 14. *Let (x^j) be a Poisson system with intensity μ on $\mathbb{R}^d, d \geq 1$. If F is in $L^1(\mathbb{R}^{kd}, \mu^{\otimes k})$, then $\mathbb{E} \left(\sum_{j_1 \neq j_2 \neq \dots \neq j_k} |F(x^{j_1}, \dots, x^{j_k})| \right) < \infty$ and*

$$\mathbb{E} \left(\sum_{j_1 \neq j_2 \neq \dots \neq j_k} G(x^{j_1}, \dots, x^{j_k}) \right) = \int_{\mathbb{R}^{kd}} F(x_1, \dots, x_k) \mu(dx_1) \dots \mu(dx_k).$$

Assume that ξ^1, \dots, ξ^k are independent symmetric α -stable Lévy processes with $\alpha \in (1 - \frac{1}{k}, 1)$ and $k \geq 2$ is an integer. Moreover, let $(\sigma_j)_{j \in \mathbb{N}}$ be a sequence of i.i.d. random variables independent of ξ^1, \dots, ξ^k and such that $\mathbb{P}(\sigma_1 = 1) = \mathbb{P}(\sigma_1 = -1)$. We then have the following.

Lemma 15. *For any $F(\cdot + \xi^1, \dots, \cdot + \xi^k) \in L^2(\mathbb{R}^k \times \Omega, \lambda_k \otimes \mathbb{P})$ the series*

$$\sum_{j_1 \neq \dots \neq j_k} \sigma_{j_1} \dots \sigma_{j_k} F(x^{j_1} + \xi^{j_1}, \dots, x^{j_k} + \xi^{j_k})$$

converges in $L^2(\Omega)$, and

$$\begin{aligned} \mathbb{E} \left(\sum_{j_1 \neq \dots \neq j_k} \sigma_{j_1} \dots \sigma_{j_k} F(x^{j_1} + \xi^{j_1}, \dots, x^{j_k} + \xi^{j_k}) \right)^2 &= \\ &= \int_{\mathbb{R}^k} \mathbb{E} \left(\sum_{\pi \in \Pi(k)} F(x_1 + \xi^1, \dots, x_k + \xi^k) \times F(x_{\pi_1} + \xi^{\pi_1}, \dots, x_{\pi_k} + \xi^{\pi_k}) \right) \\ &\quad dx_1 \dots dx_k, \end{aligned}$$

where the summation in the second integral is over all permutations π of $\{1, 2, \dots, k\}$.

This lemma follows immediately from Lemma 14 and the fact that the σ_j s are independent of ξ^1, \dots, ξ^k .

References

- [1] **Adler, R.J., Epstein, R. (1987)**, *Some limit theorems for Markov paths and some properties of Gaussian random fields*, Stochastic Process, Appl. 24, pages 157-202.

- [2] **Bai, S. , Taqqu, M.,S. (2014)**, *Generalized Hermite processes, discrete chaos and limit theorem*, Stochastic Processes and their Applications, Volume 124, April 2014, pages 1710-1739
- [3] **Bojdecki, T. ,Gorostiza, L. G. ,Talarczyk, A. (2015)**, *From intersection local time to the Rosenblatt process*, Journal of Theoretical Probability,...
- [4] **Bojdecki, T. , Talarczyk, A. (2005)**, *Particle picture approach to the self-intersection local time of the density process in $S'(\mathbb{R})$* , Stochastic Processes and their Applications, Volume 115, Issue 3, pages 449-479.
- [5] **Dobrushin, R. L., Major, P. (1979)**, *Non-central limit theorems for non-linear functionals of Gaussian fields*, Z. Wahrsch. Verw. Geb., 50 pages 27-52.
- [6] **Embrechts, P. , Maejima, M. (2002)**, *Selfsimilar Processes*, Princeton University Press
- [7] **Last, G. , Penrose, M. (2011)**, *Poisson process Fock space representation, chaos expansion and covariance inequalities*, Probability Theory and Related Fields , 150, 663-690
- [8] **Maejima, M., Tudor, C.A. (2006)**, *On the distribution of the Rosenblatt process*, Stat. Prob. Lett. 83, pages 1490-1495.
- [9] **Maejima, M., Tudor, C.A. (2012)**, *Selfsimilar processes with stationary increments in the second Wiener chaos*, Probability and Mathematical Statistics 32(1)
- [10] **Major, P. (2013)**, *Multiple Wiener-Itô Integrals*, Lecture Notes in Mathematics, Volume 849, Springer, second edition.
- [11] **Nualart, D. (2006)**, *The Malliavin Calculus and Related Topics*, Springer-Verlag Berlin, Heidelberg, second edition.
- [12] **Talarczyk, A. (2001)**, *Self-intersection local time of order k for Gaussian processes in $S'(\mathbb{R}^d)$* , Stochastic Processes and Applications, Volume 96, 2001, pages 17-72
- [13] **Taqqu, M. S. (2011)**, *Selected Works of Murray Rosenblatt*, Springer-Verlag New York, pages 29-45.
- [14] **Tudor, C. A. (2013)**, *Analysis of Variations for Self-similar Processes*, Springer International Publishing.
- [15] **Tudor, C.A. (2008)**, *Analysis of the Rosenblatt process*, ESAIM: Probability and Statistics 12, pages 230-257